Journal of Engineering Mathematics, Vol. 5, No. 1, January 1971
Wolters-Noordhoff Publishing–Groningen
Printed in the Netherlands

The Effect of Compression Work on Free Convection in Gases

H. K. KUIKEN*

Department of Mechanical Engineering, University of British Columbia, Vancouver 8, B.C., Canada

(Received March 24, 1970)

SUMMARY

This paper attempts to investigate the influence of compression work on free convective flows in gases. Since this effect is usually small a dimensional analysis using the complete set of governing equations is presented first, in order to recognize the cases for which compression work is not negligibly small in comparison with other effects. Among other things it is shown that in gases viscous heating is always of a much lower order of magnitude than compression work. Next a case of boundary layer flow that possesses similarity is studied in some detail. For various degrees of importance of compression work a numerical integration of the governing ordinary differential equations is carried out. For very strong compression work an asymptotic solution can be found. In all cases the boundary layer shows oscillatory behaviour of both the temperature and velocity profile.

1. Introduction

Since the equations governing heat transfer through moving fluids are of considerable complexity, one always has to consider approximate representations of these equations. The type of approximation depends upon the particular problem one wants to investigate. This has led to the introduction of such distinctions as, boundary layer flows, creeping flows, incompressible flows, etc. By retaining only the dominant terms pertaining to a certain class of problems one obtains the approximate equations of that class.

A problem of natural convection usually involves the observation that the Boussinesq approximation applies. This means that density variations are taken into account only insofar as these lead to the effect of buoyancy. Elsewhere one may neglect these variations, so that the fluid is considered as essentially incompressible. When stating the energy equation it is usually remarked that viscous dissipation is negligible. One also neglects, tacitly, the compression work, possibly because the fluid is considered as incompressible in the Boussinesq sense. Up to now no systematic investigation into the importance of compression work in free convection has been done. On the other hand, viscous dissipation has received the attention of some authors (Gebhart [1], Gebhart *et al.* [2], Ostrach [3]). Viscous dissipation naturally adds heat to the system and thus it will accelerate the free convective flow, and as a consequence it will increase viscous dissipation. It is somehow difficult to comprehend that the result of a certain phenomenon is to increase its own effect. This is the very reason why in this paper the influence of compression work is investigated. Indeed, whereas viscous dissipation can affect the fluid in one way only, namely by heating it, the compression work term can both heat or cool the fluid depending on whether the fluid expands or contracts.

In the case of a locally heated fluid, i.e. for convection in an upward direction, the fluid will expand due to pressure drop, which represents a cooling effect. If this effect is stronger than that of viscous dissipation, which we will prove it is, we will not have to contend with the unnatural case of a fluid flow that is accelerated by its own action. But apart from this it is interesting to introduce a term in the energy equation that absorbs heat. In the equations of free convection that are usually considered such a term does not occur. Thus, if a certain amount of heat is added to the system it will always remain in the system. This can also lead to unnatural effects. Let us take, for example, the case of a buoyant plume above a wire that is producing heat at a constant rate. It is known (Zel'dovitch [4], Yih [5], Fujii [6]) that the temperature above

* Present address: Philips Research Laboratories, Eindhoven, the Netherlands.

the wire decays to zero at a distance sufficiently far above the source. The upward velocity, however, increases indefinitely in the same direction. Apparently, this is due to the fact that a non-vanishing amount of heat has to be transported in an upward direction. When a mechanism that absorbs heat would be represented in the equations, such a result would not have been obtained.

If, on the other hand, we consider a locally cooled fluid, which results in a flow in downward direction, the compression work will heat the fluid, thus reducing the temperature difference with the ambient fluid. In the case of a downward directed plume, heat will be added until there is no longer a heat-defect far below the heat sink.

These remarks undoubtedly demonstrate that there are strong reasons for investigating the influence of the compression work terms in the energy equation. Since gases are more compressible than liquids it seems natural to investigate the problem for gases. In order to simplify the analysis further only ideal gases will be considered. We will start with a dimensional analysis that yields the relative magnitude of the various terms in the equations. This analysis is modeled after the celebrated work of Ostrach [7]. Next, a simple example will be given that shows the effect both qualitatively and quantitatively.

2. Dimensional Analysis

The equations to be considered are

(i) The equation of continuity

$$\Delta = -\frac{u_j}{\rho}\frac{\partial \rho}{\partial x_j}, \qquad \left(\Delta = \frac{\partial u_j}{\partial x_j}\right) \tag{1}$$

(ii) The momentum equations ($i=1, 2$ for two-dimensional flow)

$$\rho u_j \frac{\partial u_i}{\partial x_j} = -\rho g \delta_{i1} - \frac{\partial p}{\partial x_i} + \bar{\mu}\nabla^2 u_i + \tfrac{1}{3}\bar{\mu}\frac{\partial \Delta}{\partial x_i} + \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\frac{\partial \bar{\mu}}{\partial x_j} - \tfrac{2}{3}\Delta\frac{\partial \bar{\mu}}{\partial x_i}, \tag{2}$$

(iii) The energy equation (for ideal gas)

$$\rho c_p u_j \frac{\partial T}{\partial x_j} - u_j \frac{\partial p}{\partial x_j} = \frac{\partial}{\partial x_j}\left(k\frac{\partial T}{\partial x_j}\right) + \tfrac{1}{2}\bar{\mu}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \tfrac{2}{3}\bar{\mu}\Delta^2, \tag{3}$$

(iv) The equation of state

$$p = R\rho T. \tag{4}$$

Here x_1 measures distance in the direction parallel to the force of gravity (opposite direction), x_2 is perpendicular to x_1, u_1 and u_2 are the velocity components in the x_1 and x_2 direction respectively, g may be the acceleration due to gravity or a centrifugal acceleration, ρ is the density, p the pressure, k the thermal conductivity and $\bar{\mu}$ the dynamic viscosity. It is important to note that the second term of (3) represents compression work, while the last two terms give rise to viscous dissipation.

Ostrach [7] has investigated these equations when the gas is locally maintained at a temperature that is slightly different from that of the ambient fluid (T_e). Thus the relative temperature difference was considered to be much less than unity

$$\varepsilon = \frac{|\Delta T|}{T_e} \ll 1. \tag{5}$$

For simplicity let us assume $\Delta T > 0$ so that we will have flow in an upward direction. Since we want to investigate certain effects that are usually small, it is necessary to repeat Ostrach's analysis partly and make appropriate modifications where needed. The dimensional analysis should lead to simple equations that account for the compression work effect. It is necessary to state clearly which effects are neglected, and their relative magnitude with respect to those retained should be small. Ostrach observes that in the heated region the changes in the pressure and the density should be proportional to ε. This author therefore introduces the following transformations

$$p = p_0[\exp(-\varepsilon_1 x_1/l) + \varepsilon\tilde{p}], \quad \varepsilon_1 = gl/(RT_0) \tag{6}$$

$$\rho = \rho_0[\exp(-\varepsilon_1 x_1/l) + \varepsilon\tilde{\rho}], \tag{7}$$

$$T = T_0[1 + \varepsilon\tilde{T}], \tag{8}$$

where $\tilde{p}$, $\tilde{\rho}$ and $\tilde{T}$ are at most of order unity and l is a length characteristic for the dimensions of the system. The subscript 0 refers to the position $x_1 = 0$. The ambient gas is considered to be of uniform temperature ($T_0 = T_e$). Since only small temperature differences are considered it is reasonable to take $\bar{\mu}$ and k as constants. In any case, small changes in $\bar{\mu}$ or k will not change the character of the flow. If in addition the velocities and lengths are rendered dimensionless with

$$u_i = U\tilde{u}_i, \tag{9}$$

$$x_i = l\tilde{x}_i \tag{10}$$

we obtain the following for the equation of continuity

$$\tilde{\Delta} = \frac{\partial \tilde{u}_j}{\partial \tilde{x}_j} = \tilde{u}_j \frac{\varepsilon_1 \delta_{j1} - \varepsilon \dfrac{\partial \tilde{\rho}}{\partial \tilde{x}_j} + O\{\max(\varepsilon_1^2, \varepsilon_1\varepsilon, \varepsilon^2)\}}{1 + O\{\max(\varepsilon, \varepsilon_1)\}}. \tag{11}$$

Since ε_1 is usually small we have that the equation of continuity can be approximated by

$$\tilde{\Delta} = 0. \tag{12}$$

It is clear that, except for very small temperature differences, the value of ε_1 is much less than that of ε, especially when l is moderate. However, on an atmospheric scale ε_1 may well be larger than ε even for moderate temperature differences.

On choosing $U^2 = gl\varepsilon$ which is known to be the appropriate choice for boundary layer flows (Ostrach [7]) we obtain for the momentum and energy equations

$$\tilde{u}_j \frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} = -\tilde{\rho}\delta_{i1} - N\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \frac{1}{G^{\frac{1}{2}}}\tilde{\nabla}^2\tilde{u}_i, \tag{13}$$

$$\tilde{u}_j \frac{\partial \tilde{T}}{\partial \tilde{x}_j} + M\tilde{u}_j\left(\delta_{j1} - \varepsilon N\frac{\partial \tilde{p}}{\partial \tilde{x}_j}\right) = \frac{1}{PG^{\frac{1}{2}}}\tilde{\nabla}^2\tilde{T} + \frac{M}{G^{\frac{1}{2}}}\frac{\varepsilon}{2}\left(\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} + \frac{\partial \tilde{u}_j}{\partial \tilde{x}_i}\right)\left(\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} + \frac{\partial \tilde{u}_j}{\partial \tilde{x}_i}\right), \tag{14}$$

where terms of order ε and order ε_1 have been neglected, except for the dissipation term, since we want to obtain an estimate of the relative magnitude of this term. N, M, G and P are given by

$$N = \frac{p_0}{\rho_0 g l}, \quad M = \frac{gl}{c_p \Delta T}, \quad G = \frac{g\varepsilon l^3}{\nu^2}, \quad P = \frac{\nu\rho_0 c_p}{k}, \tag{15}$$

where ν is the kinematic viscosity and G the Grashof number. P is the Prandtl number.

The equation of state leads to

$$\tilde{p} = \tilde{\rho} + \tilde{T}. \tag{16}$$

Since it is clear that N has a very large value, it follows that $\tilde{p}$ as defined in (6) is not of order unity. If it were, the pressure term in (13) would be dominant over all other terms in the equation. We rather have to assume that $N\tilde{p}$ is a quantity of order unity. Thus let us introduce $\bar{p} = N\tilde{p}$. To first order the equation of state (11) now becomes

$$\tilde{\rho} + \tilde{T} = 0, \tag{17}$$

which expresses that the small temperature variations are almost exactly balanced by a density change of the same order. The relative pressure change due to temperature variations is of a much lower order of magnitude. By finally introducing the boundary layer transformations $(G \gg 1)$

$$\tilde{u}_1 = u\,, \quad \tilde{u}_2 = G^{-\frac{1}{4}} v\,, \quad \tilde{x}_1 = x\,, \quad \tilde{x}_2 = G^{-\frac{1}{4}} y\,, \tag{18}$$

the free convection boundary layer equations which include the compression work are obtained

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\,, \tag{19}$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \tilde{T} - \frac{\partial \bar{p}}{\partial x} + \frac{\partial^2 u}{\partial y^2}\,, \tag{20}$$

$$\frac{\partial \bar{p}}{\partial y} = 0\,,$$

$$u \frac{\partial \tilde{T}}{\partial x} + v \frac{\partial \tilde{T}}{\partial y} + Mu = \frac{1}{P} \frac{\partial^2 \tilde{T}}{\partial y^2}\,. \tag{22}$$

Terms of $O(\varepsilon)$, $O(\varepsilon_1)$, $O(M\varepsilon)$, $O(N^{-1})$ and $O(G^{-\frac{1}{4}})$ have been neglected in deriving these equations. From the derivation it must have become obvious that the relative magnitude of the viscous dissipation terms with respect to the compression work term is of the order $O(\varepsilon)$, so that the former is indeed negligibly small with respect to the latter.

Let us now focus our attention on the term Mu in (22), which represents compression work. For positive u, i.e. for flow in an upward direction this term is responsible for absorption of heat.

It is clear that the factor M is decisive for the amount of heat that will be absorbed. Since c_p is very large ($c_p \sim 10^3$ J kg^{-1} deg K^{-1}) one can only obtain larger values of M when l is very large, when ΔT is very small or in very strong acceleration fields. Since the change of $\tilde{T}$ will be of the order $O(gl/c_p \Delta T)$ it follows from (8) that

$$T = T_0 + \Delta T + O(gl/c_p)\,, \tag{23}$$

which yields the well-known meteorological result that the equilibrium temperature of the dry atmosphere drops by 1 deg K every 100 m (constant-lapse-rate atmosphere). In the derivation of the equations we have assumed that the temperature outside the boundary layer is uniform (lapse-rate equal to zero). It seems certainly possible to maintain such a non-equilibrium condition during a large enough time for the present analysis to be meaningful. However, if the present analysis is to apply to an equilibrium atmosphere one must introduce T_e in lieu of T_0 in (8). The analysis then will change accordingly.

The case of small temperature differences seems to be of importance in removing heat far above a heat source. Indeed, the temperature in a buoyant plume decreases to zero far enough above the source while the velocity u attains very large values. Thus the heat absorbing term becomes increasingly important. Also at the outer fringes of the boundary layer, where the temperature differences are small, compression work may play an important role.

It may be in order to perform the dimensional analysis for Grashof numbers that are of order unity. In that case it is necessary to make the buoyancy term and the viscous term of the same order of magnitude. The appropriate choice of U then is (Ostrach [7])

$$U = gl^2 \varepsilon / \nu\,. \tag{24}$$

The momentum and energy equations then become

$$G \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} = \tilde{T} \delta_{i1} - \frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \tilde{\nabla}^2 \tilde{u}_i\,, \tag{25}$$

$$G\left(\tilde{u}_j \frac{\partial \tilde{T}}{\partial \tilde{x}_j} + M u_1 \right) = \frac{1}{P} \tilde{\nabla}^2 \tilde{T} + GM\varepsilon \left(\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} + \frac{\partial \tilde{u}_j}{\partial \tilde{x}_i} \right) \left(\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} + \frac{\partial \tilde{u}_j}{\partial \tilde{x}_i} \right) \tag{26}$$

and again the viscous dissipation is seen to be negligible in comparison with compression work.

3. An Illustrative Example

In this section we will illustrate the theory with a simple example. It follows from equation (15) that in the case of boundary layer flows a similarity solution can be found if ΔT is proportional to l. Thus, let us consider free convection past a semi-infinite vertical flat plate which is maintained at a temperature

$$T_w = T_0 + \alpha x\,, \tag{27}$$

where $x=0$ at the leading edge. By aid of equations (15), (5) and (27) we introduce the Grashof number based on x as $G_x = g\alpha x^4/(\nu^2 T_0)$. Through introduction of

$$u = \nu \frac{G_x^{\frac{1}{2}}}{x} f'(\eta)\,, \quad v = -\nu \frac{G_x^{\frac{1}{4}}}{x} f(\eta)\,, \quad T = T_0 + \alpha x \theta(\eta)\,, \quad \eta = \frac{y}{x} G_x^{\frac{1}{4}} \tag{28}$$

into equations (2) and (3) the following ordinary differential equations are obtained

$$f''' + ff'' - (f')^2 + \theta = 0\,, \tag{29}$$

$$\frac{1}{P}\theta'' + f\theta' - f'\theta - Mf' = 0\,, \tag{30}$$

where primes stand for differentiation with respect to η. M is given by $M = g/(c_p\alpha)$. The boundary conditions to be satisfied by f and θ are

$$f(0) = f'(0) = \theta(0) - 1 = \theta(\infty) = f'(\infty) = 0\,. \tag{31}$$

It is of course rather easy to integrate the system (29–31) using a digital computer. In this way one can study the problem most conveniently for all values of M. It is, however, interesting to investigate the equations first for large values of η. From the computer results it follows that in all cases f tends exponentially to a certain positive constant $c(M)$ as η tends to infinity. Thus for large values of η the equations may be linearized as follows

$$f''' + cf'' + \theta = 0\,, \tag{32}$$

$$\frac{1}{P}\theta'' + c\theta' - Mf' = 0\,. \tag{33}$$

Since we are dealing with gases the values of the Prandtl number will be of order unity. As the analysis of the equations (32) and (33) is simplified considerably by taking $P=1$ we will take this value of the Prandtl number in the subsequent analysis. The system now reduces to

$$f^{\mathrm{V}} + 2cf^{\mathrm{IV}} + c^2 f''' + Mf' = 0\,, \tag{34}$$

which can be solved by substitution of $f = e^{\lambda\eta}$. The five values of λ that satisfy (34) are

$$\lambda_1 = 0\,, \quad \lambda_{2,3,4,5} = -\tfrac{1}{2}\left[c \pm \{c^2 \pm 4iM^{\frac{1}{2}}\}^{\frac{1}{2}}\right]. \tag{35}$$

The first of these values naturally enables us to find solutions that tend to c as $\eta \to \infty$. If we evaluate the other values for small M, i.e. for small compression work effects, we obtain

$$\lambda_{2,3} = -c \pm \frac{iM^{\frac{1}{2}}}{c} - \frac{M}{c^3} \mp \frac{2iM^{\frac{3}{2}}}{c^5} + \dots\,, \tag{36}$$

$$\lambda_{4,5} = \pm \frac{iM^{\frac{1}{2}}}{c} + \frac{M}{c^3} \mp \frac{2iM^{\frac{3}{2}}}{c^5} + \dots \tag{37}$$

Obviously, $\lambda_{4,5}$ are unsuitable, as they lead to unbounded solutions as $\eta \to \infty$. The solutions associated with $\lambda_{2,3}$ tend to zero exponentially. However on this exponential decay a slow oscillation, the frequency of which is proportional to $M^{\frac{1}{2}}/c$, is superposed. This, perhaps, is the most important result of the present investigation. It means that the compression work term induces an oscillatory behaviour at the outer edge of a free convective boundary layer. In purely free convective flows no oscillations are encountered. The present analysis shows that inclusion of the effect associated with M leads to oscillations, even for the smallest values of M.

The behaviour of f for large values of η can now be given as (M small)

$$f \to c + e^{-c\eta(1+M/c^4+\dots)}\left[A\cos\left\{\frac{M^{\frac{1}{2}}\eta}{c}\left(1-\frac{2M}{c^4}+\dots\right)\right\}+B\sin\left\{\frac{M^{\frac{1}{2}}\eta}{c}\left(1-\frac{2M}{c^4}+\dots\right)\right\}\right]. \quad (38)$$

The values of A and B can be found by considering (38) as the main part of an outer expansion that gives the solution for f in a region where the variable $M^{\frac{1}{2}}\eta$ is of order unity. This is the region where the oscillatory behaviour of f is important. The solution in the inner region where η is the natural variable, is obtained by expanding regularly in (29–30) for small values of M. The purely free convective solution furnishes the main term of this expansion. The asymptotic behaviour of this main term, for $\eta \to \infty$, can be obtained by using (34) for $M=0$. This is easily shown to be

$$f \to c(0) + (c_1 + c_2\eta)\,e^{-c(0)\eta} + \text{higher orders} \quad (39)$$

where $c(0)$, c_1 and c_2 are obtained from numerical integration of the equations (29) and (30) for $M=0$. Evaluating (38) for small values of M gives

$$c(0) + e^{-c(0)\eta}\left[A + B\frac{M^{\frac{1}{2}}}{c(0)}\eta\right] + \text{higher orders}\,. \quad (40)$$

Thus, by choosing $A=c_1$ and $B=c(0)c_2/M^{\frac{1}{2}}$ the inner and outer solutions behave similarly to first order in an intermediate region. It should be clear from this that the oscillatory behaviour can never be obtained by considering the inner region only, i.e. by regularly expanding in (29–30). This proves that a singular perturbation technique must be applied to account for this effect. It seems that for finding higher perturbations the application of a two-variable technique of the Poincaré type is most suitable for the present problem (Cole [8]). These higher perturbations are of little importance, however, since for small values of M the oscillations are overpowered by the rapid exponential decay.

For several values of M and P numerical integrations of the pertinent differential equations have been carried out. Some important results are presented in Table 1.

The value of $\theta'(0)$ is most important in deriving the heat transfer at the wall. $f''(0)$ is used for skin-friction calculations, while $f(\infty)$ is related to the total mass flow. In Figs. 1, 1a, 2 and 2a graphs representing the upward velocity and the temperature have been given for $P=0.72$. For the larger values of M the oscillations of the temperature profile are quite conspicuous. For $M<1$ the oscillations are made visible on large scale graphs. It may be expected that the oscillations will become more important for larger values of M. Whereas for small and moderate M these are only visible at the outer edge of the boundary layer, it may be expected that for very large values of M the oscillations will dominate the entire flow field. To show this let us transform the equations (29) and (30) by introduction of

$$f(\eta) = (4\,P^3M^3)^{-\frac{1}{4}}F(\mu)\,, \quad \theta(\eta) = \vartheta(\mu)\,, \quad \mu = (\tfrac{1}{4}PM)^{\frac{1}{4}}\eta \quad (41)$$

TABLE 1

P	M	$-\theta'(0)$	$f''(0)$	$f(\infty)$
0.72	0	0.5332	0.7791	1.1838
1	0	0.5951	0.7395	1.0317
0.72	0.5	0.6315	0.7031	0.8077
1	0.5	0.7009	0.6594	0.6675
0.72	1	0.7020	0.6530	0.6232
1	1	0.7757	0.6091	0.5048
0.72	4	0.9372	0.5162	0.2876
1	4	1.0252	0.4776	0.2270
	large	$(PM/4)^{\frac{1}{4}}$	$(4PM)^{-\frac{1}{4}}$	$(4P^3M^3)^{-\frac{1}{4}}$

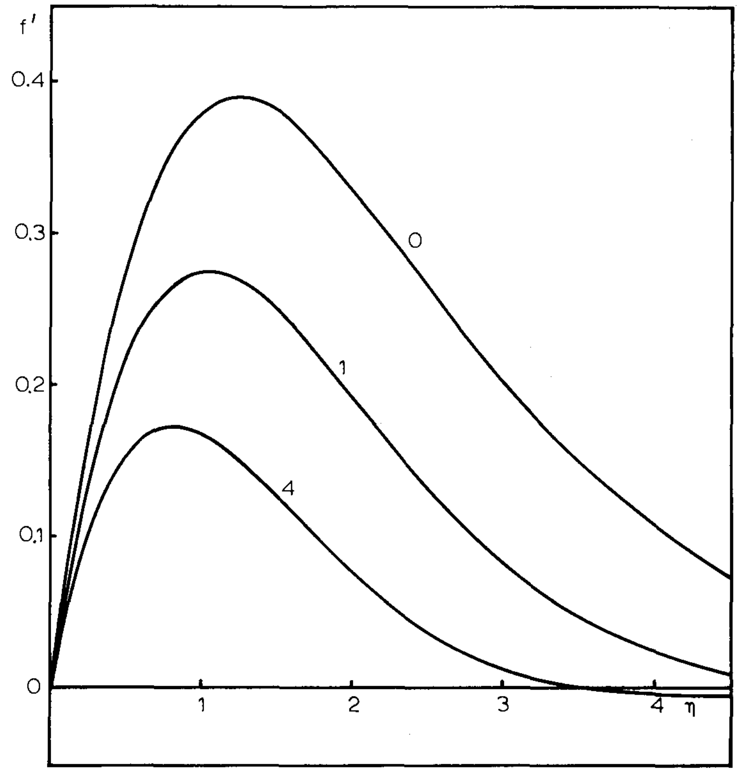

Figure 1. Dimensionless vertical velocity for $M=0$, 1 and 4.

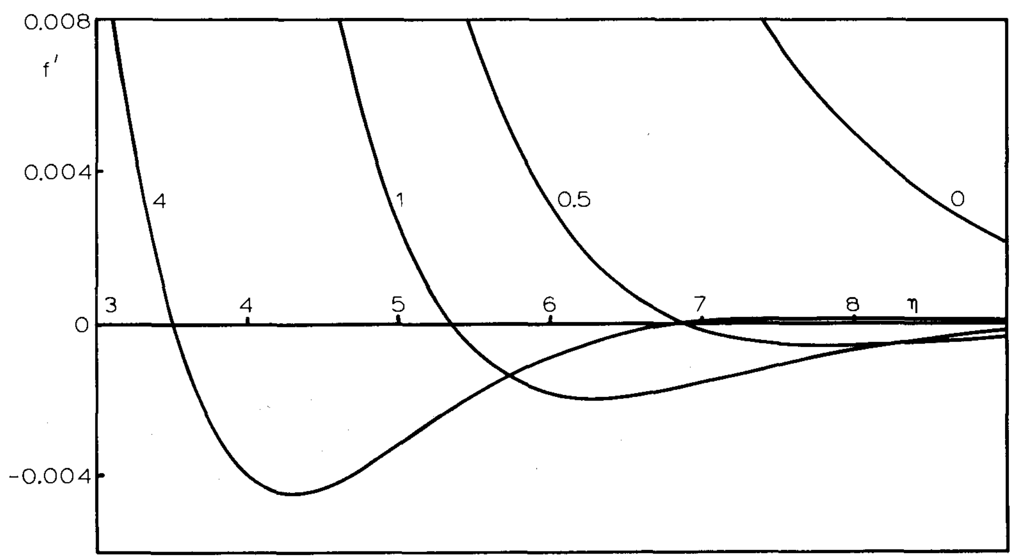

Figure 1a. Dimensionless vertical velocity in outer part of boundary layer for $M=0$, 0.5, 1 and 4.

F and ϑ satisfy the equations

$$F'''+4\vartheta+\frac{1}{PM}\{FF''-(F')^2\}=0\,, \tag{42}$$

$$\vartheta''-F'+\frac{1}{M}\{F\vartheta'-F'\vartheta\}=0 \tag{43}$$

with boundary conditions analogous to (31). Here primes stand for differentiation with respect

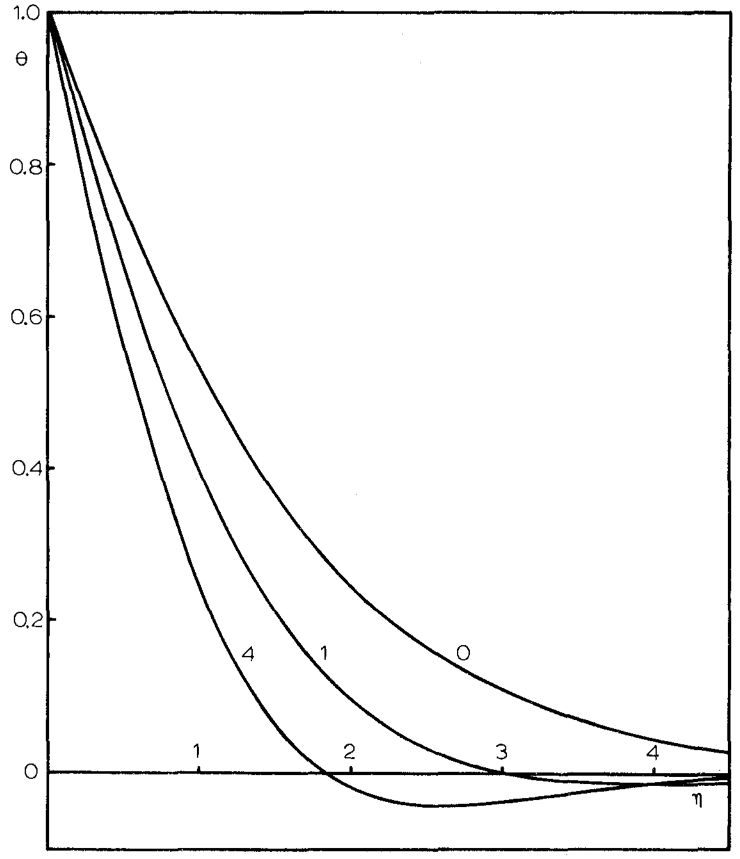

Figure 2. Dimensionless temperature for $M=0$, 1 and 4.

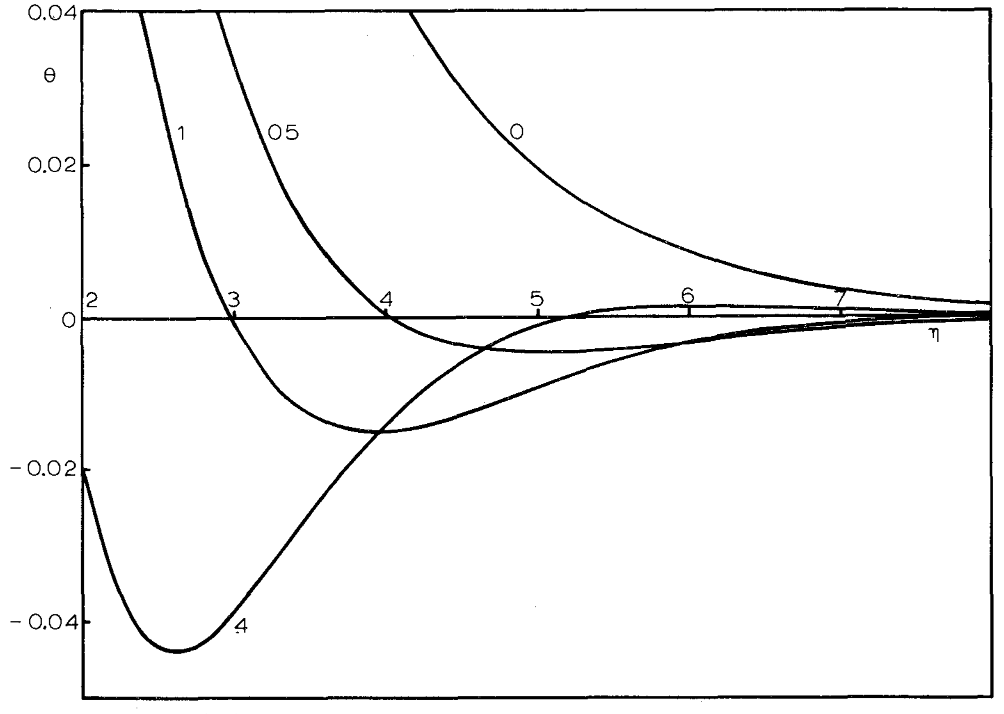

Figure 2a. Dimensionless temperature in outer part of boundary layer for $M=0$, 0.5, 1 and 4.

to μ. Since $P \sim 1$ this system is suited to an investigation for large values of M. A simple regular expansion

$$F = F_0 + \frac{1}{M} F_1 + \dots, \tag{44}$$

$$\vartheta = \vartheta_0 + \frac{1}{M} \vartheta_1 + \dots \tag{45}$$

is successful here. After some calculations the result can be obtained as follows

$$F_0 = 1 - e^{-\mu}(\sin \mu + \cos \mu), \tag{46}$$

$$F_0' = 2 e^{-\mu} \sin \mu, \tag{47}$$

$$\vartheta_0 = e^{-\mu} \cos \mu, \tag{48}$$

$$\begin{aligned} F_1 = {} & \frac{1-2P}{10P} [e^{-\mu}(3 \cos \mu - \sin \mu) - 2 e^{-2\mu} - 1] + \\ & + \frac{1+P}{4P} e^{-\mu}[-\sin \mu + \mu(\sin \mu + \cos \mu)] - \tfrac{1}{2}(1 + e^{-2\mu}) + e^{-\mu} \cos \mu, \end{aligned} \tag{49}$$

$$\begin{aligned} F_1' = {} & \frac{1-2P}{5P} [2 e^{-2\mu} - e^{-\mu}(2 \cos \mu + \sin \mu)] + \frac{1+P}{2P} e^{-\mu}(1-\mu) \sin \mu + \\ & + e^{-2\mu} - e^{-\mu}(\sin \mu + \cos \mu), \end{aligned} \tag{50}$$

$$\vartheta_1 = \frac{1-2P}{10P} [e^{-2\mu} + e^{-\mu}(2 \sin \mu - \cos \mu)] + \frac{1+P}{4P} e^{-\mu}(\sin \mu - \mu \cos \mu). \tag{51}$$

The following may serve as an explanation of these results. Let us confine ourselves to the expressions for F_0' and ϑ_0 that represent the major part of the velocity and the temperature profile respectively. Near the wall buoyancy forces the fluid to flow in an upward direction. This upward velocity cools the fluid through the action of the compression work term. As the fluid is still flowing upwards when ϑ reaches the value zero, the gas will be cooled further, so that locally a negative temperature difference with the ambient fluid will be obtained. This results in negative buoyancy which retards the flow until the velocity becomes negative. For a negative velocity the compression work term adds heat to the system so that ϑ may reach the value zero again and subsequently attain positive values. From then onwards the process starts all over again. The damping is naturally caused by viscosity. From (41) it can be seen that this process takes place over a distance that is considerably less than that needed for temperature decay in free convection flows without compression effects. Indeed, the latter is described by the variable η while for the former μ is a variable of order unity. In this range of μ the value of η is very small for large values of M.

For the heat transfer at the wall one usually introduces the local Nusselt number

$$Nu_x = -\frac{x}{\Delta T_w} \left[\frac{\partial T}{\partial y} \right]_w \tag{52}$$

Using (28) this is easily transformed into

$$\frac{Nu_x}{G_x^{\frac{1}{4}}} = -\theta'(0). \tag{53}$$

Thus the information of Table 1 can be used to determine the heat flux. For larger values of M it is convenient to use the results of the asymptotic analysis. On using (41), (48) and (51) it is easily proved that

$$-\theta'(0) = \left(\frac{PM}{4}\right)^{\frac{1}{4}} \left\{1 + \frac{2P-1}{10P}\frac{1}{M} + O\left(\frac{1}{M^2}\right)\right\}, \tag{54}$$

which shows that the heat transfer grows as M tends to infinity. This result had to be expected, since for larger M more heat will be removed from the fluid by compression work, so that more heat can flow from the wall into the fluid. For $M=4$ and $P=1$ the two-term expansion (52) predicts $-\theta'(0)=1.025$ which is remarkably close to the numerical value 1.0252. This proves that the asymptotic analysis can be used up to rather low values of M. Even for $M=1$, $P=1$ the theoretical value $-\theta'(0)=0.777$ is very close to the value found by the numerical computation.

In calculating the skin-friction a similar procedure has to be followed. It can be shown without difficulty that this is proportional to $f''(0)$. For large values of M we can find the following expression

$$f''(0) = (4PM)^{-\frac{1}{4}} \left[1 - \frac{3P+1}{20P}\frac{1}{M} + O\left(\frac{1}{M^2}\right)\right]. \tag{55}$$

This shows that the skin-friction will decrease as M tends to infinity. The following explanation may be given. For large values of M so much heat is removed from the system, that only near the wall an appreciable temperature difference with the ambient fluid exists. Thus buoyancy is only strong very near to the wall. But just here the driving force is strongly opposed by the viscous shear forces. Therefore the fluid can only reach relatively small velocities. This is in complete agreement with the earlier remark which stated that for large values of M the flow field is much closer to the wall than in the case of pure free convection

By using the knowledge acquired through the study of the similarity solution it is possible to attack with success several non-similar boundary layer problems. One could consider for example the case of uniform heat flux or the case of uniform wall temperature. It must be clear that these problems have to be treated in the same way as the problem just presented. A difference is that M will now be a function of x and thus one will have to develop coordinate expansions for both large and small values of M_x where M_x is given by

$$M_x = \frac{gx}{c_p(T_w - T_0)}. \tag{56}$$

For intermediate values of M_x there is an obvious difference between the similar and the non-similar problems. In the similar case one can find solutions in this region by integration of a set of ordinary differential equations. For the non-similar case this cannot be done. One usually finds satisfactory solutions by interpolation between the cases of large and small M. It is seen that for a uniform wall-temperature M_x is large downstream, while near the leading edge only small values of M_x can exist.

4. Concluding Remarks

In the present paper it has been shown how compression work can affect free convective flows. Since the parameter that is decisive about its importance is large for small temperature differences, the compression work seems to become most important in the final stages of a heat transfer process when due to effective convection and conduction, the temperature differences have become very small. Under such conditions its action will be to remove abundant heat from the system. From the dimensional analysis it became clear that, at least for gases, it is not realistic to study any dissipative heating when the Boussinesq approximation is applied. The relative magnitude of viscous heating with respect to compression work was shown to be of the same order of magnitude as the terms neglected in the Boussinesq approximation.

Practical applications that sustain the present work are perhaps most easily found in meteorology. It is well-known that the heating of the mountain winds that are associated with names as föhn or chinook, is caused almost exclusively by compression of the air as it descends along the slopes of the mountain. The temperature rise of 1 deg K for every 100 m descent is also given by the present work (23). This indeed proves that viscous heating does not play any role here.

In the case of boundary layer flows in the atmosphere examples can be given only for very small temperature differences. If we take $g=10$, $T_0^{-1}=1/300$, $\Delta T=0.1$, $c_p=1000$, $\nu=10^{-5}$, $l=10$, where MKSA units have been used, M will have the value 1, so that the compression work is moderately strong. Since $\varepsilon \sim 10^{-3}$, $\varepsilon_1 \sim 10^{-3}$, $N^{-1} \sim 10^{-3}$, $M\varepsilon \sim 10^{-3}$, $G^{-\frac{1}{4}} \sim 10^{-\frac{5}{2}}$, the neglected effects are certainly of very small order of magnitude. In order to show the effect in a laboratory experiment one must resort to very large artificial gravity fields, e.g. in fastly rotating containers. A practically feasible example could be $g=10^4$, $\Delta T=0.1$, $l=0.1$ which gives $M=10$, so that compression work is relatively strong. The neglected effects are again small in comparison with those retained, and boundary layer theory may be applied as the Grashof number is large. By taking $\Delta T=1$ this example gives moderate influence of compression work.

A final remark concerns the oscillations induced by the compression work. It is known that oscillatory flows are more likely to become unstable than non-oscillatory flows. Thus, although the oscillations induced by the work of compression are usually very small as compared with the complete flow, they may be instrumental in triggering off larger instabilities.

Acknowledgements

The author is indebted to the National Research Council of Canada for financial support.

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